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Non-Gaussian random fields and their multiple Markov properties

Si Si (愛知県立大学)
Faculty of Information Science and Technology
Aichi Prefectural University,
Aichi-ken 480-1198, Japan

Abstract

We have already discussed the Markov property of Gaussian random fields $X(C)$ with the parameter C running through the class $\mathbf{C}$ where the class $\mathbf{C}$ is taken to be $\mathbf{C}=\{C; C\in C^2, \text{diffeomorphic to } \mathrm{S}^1\}$. In this paper we introduce the Markov property for non-Gaussian random field $X(C)$, namely a random field of homogeneous chaos.

1 Introduction

When we discuss the analysis of a stochastic process $X(t)$ and a random field $X(C)$, we usually consider operations acting on the values of $X(t)$ or $X(C)$; for example the best predictor is formed by taking a function, in general nonlinear function, of the observed values like $\varphi(X(t), t\in T)$ of $X(t)$ and $\varphi(X(C), C\in\mathbf{C})$ of random fields $X(C)$.

In order to establish the analysis of those functions, we propose to take the *innovation* of $X(t)$ or that of $X(C)$ and express the original random functions as functionals of the innovation, so that they are ready to be analyzed. Often, actually in many favorable cases, the innovation is taken to be a *white noise*. Thus, the white noise analysis has become a basic tool for the investigation of stochastic processes and random fields.

With this innovation approach in mind, we aim at defining multiple Markov properties for some non Gaussian processes or fields and studying their characterization. This notion is a generalization of the Markov properties established for Gaussian process, although we need a new idea for the definition. There, we remind that as in the case of a Gaussian process, the innovation will play a dominant role.

In what follows some background will be prepared, and see the significance of taking random fields compared to stochastic processes from the viewpoint of information theory and the theory of dependency of random complex phenomena. Needless to say, we always keep computability in mind.

2 Operations acting on processes and fields

Various kinds of operations on stochastic processes and random fields help us in the investigation of the way of dependency of those random phenomena in question, in particular multiple Markov properties.

Linear and nonlinear functions.

Assume that $X(t)$'s are Gaussian in distribution. It is not so easy to analyse functions $\varphi(X(t), t \in T)$ of those Gaussian variables directly. The most useful method is, as was mentioned before, that we first form the innovation of $X(t)$, which is always taken to be white noise, of a given process discussed in this paper, then functions of $X(t)$'s can be rephrased as functionals of the innovation. Now, one is ready to analyze those functionals. Note that the causality always holds in this case and that every operation acting on those Gaussian functionals is linear. In such a case, it is easy to check if the quantity in question is computable.

It is well known that for a weakly stationary stochastic processes the Fourier analysis plays an important role (see the Wiener theory). In particular, the spectrum is one of the characteristics of stationary stochastic processes. Also in this case, only linear transformations of the process $X(t)$ or $X(C)$ are involved; namely it is easily checked if the given quantity is computable or not.

Our aim is to discuss non-Gaussian case, so that we need a review of some known results and further background.

Transformations of time parameter

Another kind of operation for $X(t)$ and $X(C)$ is change of time parameter t and of space-time parameter C, respectively.

(1) Time shift

The simplest and in fact most important example is the time shift $T_t : s \to s + t$:

$$T_t X(s) = X(s+t). \tag{2.1}$$

To fix the idea let us take $X(t)$ to be a white noise which is now viewed as the time derivative of a Brownian motion $B(t)$; the $\dot{B}(t)$ is a white noise. In this case we call $\{T_t, t \in R\}$, after S. Kakutani, the *flow of Brownian motion.*

This flow describes the evolution of the random pheneomena that are represented as functionals of white noise. More precisely, let $\varphi(\dot{B})$ be a white noise functional in Hilbert space (L^2) which contains white noise functionals with finite variance, then time evolution is expressed in terms of the untary operator U_t as

$$U_t \varphi(\dot{B}) = \varphi(T_t \dot{B}) \tag{2.2}$$

We can therefore speak of the spectral multiplicity of the unitary group $\{U_t\}$ on a subspace of (L^2), the choice of which depends on the problem to be discussed. For instance, if the entire space (L^2) is concerned, the multiplicity is infinite, while the subspace spanned by linear functionals of white noise has unit multiplicity. Computability necessarily requires finite multiplicity.

For the case of a stochastic process with higher dimensional parameter, say R^d parameter, the notion of multiplicity can also be introduced by taking the course of propagation to be the radial direction. The following assertion is almost obvious.

Let $\{X(a), a \in R^d\}$ be Lévy's Brownian motion in a sense that

1) it is Gaussian,

2) $X(a) - X(b)$ has mean 0 and variance $|a - b|$,

3) $X(O) = 0$, where O is the origin.

Proposition *Lévy's Brownian motion has infinite multiplicity.*

As a result, we claim that finitely many channels can not send the full information contained in Lévy's Brownian motion.

If we wish to describe finer way of dependency, then multiple Markov properties in a weak sense, defined like in the Gaussian case, can be defined in each cyclic subspace.

(2) The whiskers as time change operators

White noise analysis has an aspect of infinite dimensional harmonic analysis since the white noise measure μ is invariant under the infinite dimensional rotation group. It has significant one-parameter subgroups which come from one-parameter families of diffeomorphism of the parameter space. They are called *whiskers*. The shift discussed above is a good example of a whisker. We can say that the isotropic dilation, which is another whisker, has infinitely many cyclic subspaces, that is they have similar spectral properties to the shift. The same is true for the special conformal transformation.

Such an observation is helpful when whiskers that come from the conformal tarnsformations acting on the parameter space are discussed in connection with the computability. We emphasize the significance of the role of whiskers when we form the innovation from the variation of random field, which leads us to define multiple Markov property with the help of innovation.

(3) Random time parameter

The so-called *subordination* is an interesting operation for a stochastic process $X(t)$. The time parameter t is replaced by an increasing additive process $Y(t)$. The case where $Y(t)$ is taken to be an increasing stable process with exponent α, with $0 < \alpha < 1$, has been discussed. (This topic was discussed in [3] from the viewpoint of the information theory.)

If we consider computability, we may discuss the case where $Y(t)$ is taken to be a Poisson process, so that $X(t)$ is observed at random times that appear successively with exponential holding time. If, in particular, $X(t)$ is taken to be a Brownian motion, we can see the probability distribution of the subordinated process $X(Y(t))$. Subordination is an interesting topic, but we do not go into details since there is no direct connection with Markov property, which is our main topic.

It is noted that the facts discussed in this section are quite different from those discussed in Section 1 from the viewpoint of computation.

3 Information theoretical study

We now come to a random field $X(C)$ indexed by a manifold C in R^d, the parameter space of white noise. Before we come to the study of information theoretical properties of $X(C)$, we wish to emphasize the significant advantages of taking a field $X(C)$ instead of a process $X(t), t \in R^d$, as a mathematical model of random complex systems. Namely,

A) $X(C)$ carries more information than $X(t)$ when the parameter moves.

We know that $X(t)$ for $d = 1$ has usually unit multiplicity but $X(C)$ has infinite multiplicity except degenerated cases. Further, we see from an intuitive observation that t is 0-dimensional and runs through a finite dimensional space, while the dimension of C is at least one and moves in an infinite dimensional space $\mathbf{C}$. It means that $X(C)$ expresses more complex random phenomena than $X(t)$.

B) For C we have complex ways of deformation. In some important examples (e.g. [3]) the innovation of the field can be obtained by deformations of C (including the mappings from C onto itself). As a result, various ways of dependence can be discussed, in particular multiple Markov properties.

Let us have a quick review of the representation theory of a Gaussian process $X(t)$ and give a remark so that we extend the results to random fields which are either Gaussian or non Gaussian expressed as a homogeneous chaos. A representation of $X(t)$ in terms of a white noise $\dot{B}(t)$ such that

$$X(t) = \int_0^t F(t,u)\dot{B}(u)du, \tag{3.1}$$

is called *canonical* if the conditional expectation $E[X(t)|X(u), u \leq s]$ with $s \leq t$ is given by

$$\int_0^s F(t,u)\dot{B}(u)du. \tag{3.2}$$

In such a case, we can obtain the innovation $\dot{B}(t)$ by a causal and linear operator; formally speaking by the inverse of the integral operator F. The idea behind such an observation is that we define a stochastic process as a random function that gains a new information (actually, expressed by the innovation) at each instant. The probabilistic structure of a process is determined by the variation of the field involving innovation and, of course, past values.

If $X(t)$ is a nonlinear function of white noise, then it is no more Gaussian and we can not say that innovation is obtained in a similar manner, as is easily understood (see 1 in the next section).

However, if a random field $X(C)$ formed by a homogeneous chaos in the form

$$X(C) = \int_{(C)^n} F(C, u_1, u_2, ..., u_n) : x(u_1)x(u_2)...x(u_n) : du^n, \tag{3.3}$$

where (C) is the domain enclosed by C.

Then the innovation can be obtained from $\delta X(C)$ by using nonlinear operations (see [2]). There we have assumed that the kernel F is the canonical kernel.

In order to carry on the calculus, we have to assume that C runs through a certain class of smooth ovaloids.

With this remark we are now able to define a generalization of multiple Markov properties.

4 Multiple Markov properties for non-Gaussian case

For a Gaussian case, the N-ple Markov property can be characterized by the canonical kernel; namely it should be a Goursat kernel of order N.

We now think of the non-Gaussian case which is restricted to be a homogeneous chaos.

1. Stochastic process $X(t)$

Suppose that $X(t)$ is not a Gaussian process, say homogeneous chaos (of white noise) of order greater than 1. Then it is impossible to obtain the innovation $\dot{B}(t)$ from the variation $\delta X(t)$. For example, let

$$X(t) = \int_0^t \int_0^t F(t; u_1, u_2) : x(u_1)x(u_2) : du_1 du_2 \tag{4.1}$$

be given. Then its variation is

$$\delta X(t) = dX(t) = dt \int_0^t \int_0^t \frac{\partial}{\partial t} F(t, u_1, u_2) : x(u_1)x(u_2) : du_1 du_2 + 2dt x(t) \int_0^t F(t, t, u_2) x(u_2) du_2.$$

Here we see that the second term has a different order, but $\int_0^t F(t, t, u_2)x(u_2)du_2$ is not a conditional expectation of $X(t + dt)$. since it is orthogonal to $X(s), s \leq t$; hence not a function of the $X(s)$.

2. Random field $X(C)$

We now come to the case of random field which is not Gaussian but homogeneous chaos. To fix the idea let us consider the case of quadratic chaos (take $n = 2$ for $X(C)$ in the last section),

$$X(C) = \int_{(C)} F(C; u_1, u_2) : x(u_1)x(u_2) : du_1 du_2. \tag{4.2}$$

Assume that F is the canonical kernel. Note that canonical property can be defined in the same manner to the case of Gaussian random field. Of course the uniqueness of the canonical kernel is guaranteed.

From the result in the paper [2] we can obtain the innovation $\{x(s), s \in C\}$ from $\delta X(C)$ and also the conditional expectation $E[X(C)|\mathrm{B}(C')]$, where $\mathrm{B}(C')$ is the sigma-field of events determined by $X(C''), C'' < C'$. Here $C'' < C'$ means that C'' is inside of C'. There is an important fact; namely, the conditional expectation is a nonlinear function of the $X(C'')$ with $C'' < C'$ (note that no more Gaussian case). It is therefore a nonlinear function, in reality a quadratic function, of the $x(u), u \in (C')$.

Since the conditional expectation is the projection to the space spanned by the non linear functions of $X(C'')$, $C'' < C'$. Hence, it is the projection to nonlinear function of $x(u), u \in (C')$. Thus we have

$$E[X(C)|\mathrm{B}_{C'}(X)] = \int_{(C')} F(C, u_1, u_2) : x(u_1)x(u_2) : du_1 du_2. \tag{4.3}$$

The results obtained so far hold for a homogeneous chaos of any order.

Thus, multiple Markov properties for $X(C)$ of homogeneous chaos can be defined in a similar manner to the case of Gaussian fields since conditional expectation is formed by the innovation. To make sure, we give

Definition Let $X(C)$ be given by

$$X(C) = \int_{(C)^n} F(C, u_1, u_2, ..., u_n) : x(u_1)x(u_2) \cdots x(u_n) : du^n \tag{4.4}$$

with a canonical kernel F. For any choice of C_i's such that $C_0 \le C_1 < ... < C_N < C_{N+1}$,

1. $E[X(C_i)|\mathrm{B}_{C_0}(X)], i = 1, 2, ..., N$, are linearly independent and
2. $E[X(C_i)|\mathrm{B}_{C_0}(X)], i = 1, 2, ...N + 1$ are linearly dependent

then, $X(C)$ is said to be *N-ple Markov.*

Theorem *If a random field $X(C)$ of homogeneous chaos is N-ple Markov, then its canonical kernel is a Goursat kernel of order N.*

Proof. For proof we only note that the conditional expectation is a nonlinear function of the known values, unlike Gaussian case. For the rest of the proof we can follow the method given in [4].

Corolary *The predictor of an N-ple Markov random field of homogeneous chaos is computable. More precisely, the best predictor is a linear combination of the random variable obtained from the values of the past.*

To close this paper, we note that the multiple Markov properties indicate not only way of dependency, but also suggest computability of the best predictor.

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